ABOUT THE CATEGORIES OF
SEGAL TOPOLOGICAL ALGEBRAS

MART ABEL

Tallinn University and University of Tartu

ESTONIA
Definition 1. We say that a topological algebra \((A, \tau_A)\) is a left (right or two-sided) Segal topological algebra in a topological algebra \((B, \tau_B)\) via an algebra homomorphism \(f : A \to B\), if

1) \(\text{cl}_B(f(A)) = B\);
2) \(\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}\), i.e., \(f\) is continuous;
3) \(f(A)\) is a left (respectively, right or two-sided) ideal of \(B\).
Definition 1. We say that a topological algebra \((A, \tau_A)\) is a left (right or two-sided) Segal topological algebra in a topological algebra \((B, \tau_B)\) via an algebra homomorphism \(f : A \to B\), if

1) \(\text{cl}_B(f(A)) = B\);
2) \(\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}\), i.e., \(f\) is continuous;
3) \(f(A)\) is a left (respectively, right or two-sided) ideal of \(B\).

In what follows, a Segal topological algebra will be denoted shortly by a triple \((A, f, B)\) and could be left, right or two-sided.
The category $S(B)$ of Segal topological algebras
The category $S(B)$ of Segal topological algebras

Fix any topological algebra $(B, \tau_B)$.
The category $S(B)$ of Segal topological algebras

Fix any topological algebra $(B, \tau_B)$.

The objects of the category $S(B)$ are all Segal topological algebras in the topological algebra $B$, i.e., all Segal algebras in the form of triples $(A, f, B), (C, g, B), ....$
The category $S(B)$ of Segal topological algebras

Fix any topological algebra $(B, \tau_B)$.

The objects of the category $S(B)$ are all Segal topological algebras in the topological algebra $B$, i.e., all Segal algebras in the form of triples $(A, f, B), (C, g, B), \ldots$.

The morphisms between Segal topological algebras $(A, f, B)$ and $(C, g, B)$ are all continuous algebra homomorphisms $\alpha : A \to C$, satisfying $g(\alpha(a)) = (1_B \circ f)(a) = f(a)$ for every $a \in A$, i.e., making the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{g} & B \\
\end{array}
\]

commutative.
The composition of morphisms of $S(B)$ is defined componentwise as follows:
The composition of morphisms of \( S(B) \) is defined componentwise as follows:

for any \((A, f, B), (C, g, B), (E, h, B) \in \text{Ob}(\textbf{Seg})\) and arbitrary algebra homomorphisms \( \alpha : A \to C \), \( \gamma : C \to E \) such that \( g \circ \alpha = f \) and \( h \circ \gamma = g \), the composition of \( \gamma \) and \( \alpha \) is \( \gamma \circ \alpha \)

\[
\begin{align*}
A & \xrightarrow{f} B \\
\alpha \downarrow & \downarrow 1_B \\
C & \xrightarrow{g} B \\
\gamma \downarrow & \downarrow 1_B \\
E & \xrightarrow{h} B
\end{align*}
\]

\[
h \circ (\gamma \circ \alpha) = f.
\]
The category \textbf{Seg} of Segal topological algebras
The category $\text{Seg}$ of Segal topological algebras

The objects of the category $\text{Seg}$ are all Segal topological algebras.
The category $\text{Seg}$ of Segal topological algebras

The objects of the category $\text{Seg}$ are all Segal topological algebras. The morphisms between Segal topological algebras $(A, f, B)$ and $(C, g, D)$ are all such pairs $(\alpha, \beta)$ of continuous algebra homomorphisms $\alpha : A \to C$ and $\beta : B \to D$, for which $g \circ \alpha = \beta \circ f$. 
The category $\text{Seg}$ of Segal topological algebras

The objects of the category $\text{Seg}$ are all Segal topological algebras. The morphisms between Segal topological algebras $(A, f, B)$ and $(C, g, D)$ are all such pairs $(\alpha, \beta)$ of continuous algebra homomorphisms $\alpha : A \to C$ and $\beta : B \to D$, for which $g \circ \alpha = \beta \circ f$.

Hence, in case $(A, f, B), (C, g, D) \in \text{Ob}(\text{Seg})$ and $(\alpha, \beta) \in \text{Mor}((A, f, B), (C, g, D))$, we have a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\alpha} & & \downarrow{\beta} \\
C & \xrightarrow{g} & D
\end{array}
$$
The composition of morphisms of $\text{Seg}$ is defined componentwise as follows:
The composition of morphisms of \( \text{Seg} \) is defined componentwise as follows:

for any \( (A, f, B), (C, g, D), (E, h, F) \in \text{Ob}(\text{Seg}) \) and arbitrary morphisms \( (\alpha, \beta) : (A, f, B) \to (C, g, D), (\gamma, \delta) : (C, g, D) \to (E, h, F) \), the composition of \( (\gamma, \delta) \) and \( (\alpha, \beta) \) is \( (\gamma, \delta) \circ (\alpha, \beta) = (\gamma \circ \alpha, \delta \circ \beta) \).

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \alpha & & \downarrow \beta \\
C & \xrightarrow{g} & D \\
\downarrow \gamma & & \downarrow \delta \\
E & \xrightarrow{h} & F
\end{array}
\]

\[
h \circ (\gamma \circ \alpha) = (\delta \circ \beta) \circ f.
\]
An object $O$ of the category $C$ is called
An object $O$ of the category $\mathcal{C}$ is called
a) the **initial object** of the category $\mathcal{C}$ if for any object $C \in \mathcal{C}$ the set $\text{Mor}(O, C)$ consists of exactly one morphism;
An object $O$ of the category $C$ is called
a) the initial object of the category $C$ if for any object $C \in C$ the set $\text{Mor}(O, C)$ consists of exactly one morphism;
b) the terminal object of the category $C$ if for any object $C \in C$ the set $\text{Mor}(C, O)$ consists of exactly one morphism;
An object $O$ of the category $C$ is called
a) the initial object of the category $C$ if for any object $C \in C$ the set $\text{Mor}(O, C)$ consists of exactly one morphism;
b) the terminal object of the category $C$ if for any object $C \in C$ the set $\text{Mor}(C, O)$ consists of exactly one morphism;
c) the zero object of the category $C$ if $O$ is both initial and terminal object of the category $C$. 
Suppose that we have two objects $A, B$ of category $\mathcal{C}$ and two morphisms $f, g \in \text{Mor}(A, B)$.

d) The equalizer of morphisms $f$ and $g$ is the pair $(O, m)$, where $O \in \text{Ob}(\mathcal{C}), m \in \text{Mor}(O, A)$ such that $f \circ m = g \circ m$ and for every pair $(X, q)$ with $X \in \text{Ob}(\mathcal{C}), q \in \text{Mor}(X, A)$ and $f \circ q = g \circ q$, there exists a unique $p \in \text{Mor}(X, O)$ such that $q = m \circ p$;
Suppose that we have two objects $A, B$ of category $\mathcal{C}$ and two morphisms $f, g \in \text{Mor}(A, B)$.

d) The equalizer of morphisms $f$ and $g$ is the pair $(O, m)$, where $O \in \text{Ob}(\mathcal{C})$, $m \in \text{Mor}(O, A)$ such that $f \circ m = g \circ m$ and for every pair $(X, q)$ with $X \in \text{Ob}(\mathcal{C})$, $q \in \text{Mor}(X, A)$ and $f \circ q = g \circ q$, there exists a unique $p \in \text{Mor}(X, O)$ such that $q = m \circ p$;

\[
\begin{array}{ccc}
O & \xrightarrow{m} & A & \xrightarrow{f} & B & \xrightarrow{n} & P \\
\uparrow{p} & & \downarrow{q} & \xrightarrow{g} & \downarrow{r} & \downarrow{s} & \\
X & & & & & & Y
\end{array}
\]

e) the coequalizer of morphisms $f$ and $g$ is the pair $(P, n)$, where $P \in \text{Ob}(\mathcal{C})$, $n \in \text{Mor}(B, P)$ such that $n \circ f = n \circ g$ and for every pair $(Y, r)$ with $Y \in \text{Ob}(\mathcal{C})$, $r \in \text{Mor}(B, Y)$ and $r \circ f = r \circ g$, there exists a unique $s \in \text{Mor}(P, Y)$ such that $r = s \circ n$. 
The **equalizer** of morphisms $\alpha, \beta \in \text{Mor}((A, f, B), (C, g, B))$ is a pair $((E, h, B); \epsilon)$ such that

1) $(E, h, B) \in \text{Ob}(S(B))$ and $\epsilon \in \text{Mor}((E, h, B), (A, f, B))$ with $\alpha \circ \epsilon = \beta \circ \epsilon$;

2) for any pair $((D, j, B); \delta)$ with $(D, j, B) \in \text{Ob}(S(B))$ and $\delta \in \text{Mor}((D, j, B), (A, f, B))$ with $\alpha \circ \delta = \beta \circ \delta$, there exists unique $\gamma \in \text{Mor}((D, j, B), (E, h, B))$ with $\epsilon \circ \gamma = \delta$:

```
D ----> j ----> B
|        |        |
|        |        |
|        |        |
|        |        |
|        |        |
E ----> h ----> B
|        |        |
|        |        |
|        |        |
|        |        |
|        |        |
A ----> f ----> B
|        |        |
|        |        |
|        |        |
|        |        |
|        |        |
C ----> g ----> B
```
Let $A, B, C$ be any objects of a category $C$ and $f : B \to A, g : C \to A$ any morphisms in $C$.

f) The pullback of morphisms $f$ and $g$ is an ordered triple $(P, \alpha, \beta)$ such that $P$ is an object of $C$, $\alpha : P \to B, \beta : P \to C$ are the morphisms in $C$ such that $f \circ \alpha = g \circ \beta$ and for every ordered triple $(Y, \gamma, \delta)$ with $Y$ and object of $C$, $\gamma : Y \to B, \delta : Y \to C$ morphisms in $C$ with $f \circ \gamma = g \circ \delta$, there exists a unique morphism $\epsilon : Y \to P$ such that $\gamma = \alpha \circ \epsilon$ and $\delta = \beta \circ \epsilon$.
Let $A$, $B$, $C$ be any objects of a category $C$ and $f : A \to B$, $g : A \to C$ any morphisms in $C$.

g) The pushout of morphisms $f$ and $g$ is an ordered triple $(P, \alpha, \beta)$ such that $P$ is an object of $C$, $\alpha : B \to P$, $\beta : C \to P$ are the morphisms in $C$ such that $\alpha \circ f = \beta \circ g$ and for every ordered triple $(Y, \gamma, \delta)$ with $Y$ and object of $C$, $\gamma : B \to Y$, $\delta : C \to Y$ morphisms in $C$ with $\gamma \circ f = \delta \circ g$, there exists a unique morphism $\epsilon : P \to Y$ such that $\gamma = \epsilon \circ \alpha$ and $\delta = \epsilon \circ \beta$.
Let \((A, f, B), (C, g, D), (E, h, F)\) be any objects of \textbf{Seg}, \((\alpha, \beta) \in \text{Mor}((E, h, F), (A, f, B)), (\gamma, \delta) \in \text{Mor}((E, h, F), (C, g, D))\). An object \((Q, k, R)\) of the category \textbf{Seg}, together with morphisms \((\kappa, \lambda) \in \text{Mor}((A, f, B), (Q, k, R)), (\mu, \nu) \in \text{Mor}((C, g, D), (Q, k, R))\), is the pushout of morphisms \((\alpha, \beta)\) and \((\gamma, \delta)\), if

1) \((\kappa, \lambda) \circ (\alpha, \beta) = (\mu, \nu) \circ (\gamma, \delta)\);

2) for every \((S, l, T) \in \text{Ob}((\textbf{Seg})\), \((\sigma, \rho) \in \text{Mor}((A, f, B), (S, l, T)), (\omega, \phi) \in \text{Mor}((C, g, D), (S, l, T))\) with \((\sigma, \rho) \circ (\alpha, \beta) = (\omega, \phi) \circ (\gamma, \delta)\), there exists unique \((\psi, \xi) \in \text{Mor}((Q, k, R), (S, l, T))\) such that \((\psi, \xi) \circ (\kappa, \lambda) = (\sigma, \rho)\) and \((\psi, \xi) \circ (\mu, \nu) = (\omega, \phi)\).
### Overview of the progress so far

<table>
<thead>
<tr>
<th></th>
<th>$S(B)$</th>
<th>Seg</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Initial object</strong></td>
<td>NASC</td>
<td>Always exists</td>
</tr>
<tr>
<td><strong>Terminal object</strong></td>
<td>NASC</td>
<td>Always exists</td>
</tr>
<tr>
<td><strong>Zero object</strong></td>
<td>NASC</td>
<td>Always exists</td>
</tr>
<tr>
<td><strong>Equalizer</strong></td>
<td>NASC</td>
<td>SC</td>
</tr>
<tr>
<td><strong>Coequalizer</strong></td>
<td>Always exists</td>
<td>Always exists</td>
</tr>
<tr>
<td><strong>Pullback</strong></td>
<td>NASC</td>
<td>SC</td>
</tr>
<tr>
<td><strong>Pushout</strong></td>
<td>Always exists</td>
<td>SC</td>
</tr>
<tr>
<td><strong>Product</strong></td>
<td>NASC</td>
<td>?</td>
</tr>
<tr>
<td><strong>Coproduct</strong></td>
<td>Always exists</td>
<td>SC</td>
</tr>
<tr>
<td><strong>Limit</strong></td>
<td>SC (for at most countable case)</td>
<td>?</td>
</tr>
<tr>
<td><strong>Colimit</strong></td>
<td>Always exists</td>
<td>?</td>
</tr>
</tbody>
</table>

NASC - necessary and sufficient conditions  
SC - sufficient conditions
References


13. Abel, M. *Pushouts in the category* $\text{Seg}$ *of Segal topological algebras.* Finished manuscript for the Proceedings of ICTAA 2021.